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RESEARCH

Parameterizing the Curvilinear Roofs of Traditional Chinese Architecture

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Abstract

The curvilinear roofs of ancient Chinese wooden architecture feature unique functional, hierarchical, and aesthetic elements. However, the digitalization of these roofs has yet to be explored because of the lack of accurate mathematical models for these roof geometries. This research aims to parameterize the design and construction rules for these roofs as stipulated in the *Yingzao fashi*. We first illustrate the major features of curvilinear roofs, followed by a comprehensive investigation on the raising and bending rules via the original texts and drawing samples provided in the *Yingzao fashi* and the *Annotated yingzao fashi* by Sicheng Liang. Then, the mathematical models are developed. The main contribution of this research is the transformation of the design and building rules for complex curvilinear roofs into mathematical models, enabling designers to generate such roof architecture and adding new insights to existing mathematical methods in studying ancient architecture.

Keywords Parametric design · Iterative algorithm · Traditional architecture · Logic of design · Roof architecture · Chinese architecture

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Introduction

> *Order is the balanced adjustment of the details of the work separately, and, as to the whole, the arrangement of the proportion with a view to a symmetrical result.*
>
> Vitruvius I.II.2 (1931–1934: I, 25)

In Western architecture, just as musical intervals are a set of ratios of integers, architects use such ratios of integers to determine the scales and proportions of space and form. Like many of his contemporaries, Vitruvius argued that the proportionate dimensions of the human body should also be used in building operations: the height of the Doric column should be seven times the diameter, in the Ionic, it should be nine, etc. The influence of mathematics on architecture was emphasized by Renaissance architects such as Leon Battista Alberti, Leonardo da Vinci, and Andrea Palladio, who deemed the laws of proportion to be the nature of the universe. The theoretical position they occupied was challenged in the eighteenth century, but contemporary architects have never stopped attempting to establish a "modulus system." For instance, Le Corbusier built his ingenious "Modulor" system by reconstructing the Fibonacci Sequence (Corbusier 1954). The Dutch monk and architect Hans van der Laan developed his Plastic Number theory, which could generate the Padovan sequence (Padovan 2015). Since the Modernist Movement, architects have been hoping to achieve the goal of "making the good easy and the bad difficult" by adopting some modulus system capable of precisely matching Corbusier's expectations. Thus, the generation of a particular sequence by iteration is a design tool that has been used effectively to design architectural forms and spaces.

Similar concepts of scales and proportions have existed in traditional Chinese architecture for thousands of years, and the associated modular units or systems can be frequently found in classic Chinese architecture literature, such as the *Yingzao fashi* (dated to 1100) and *Gongcheng Zuofa Zeli* (published in 1734). In particular, the *Yingzao fashi* (Treatise on Architectural Methods), introduces general rules and regulations for building design and construction, followed by exemplary specifications and illustrations. These rules and regulations, consisting of various scales and proportions, formed the fundamental modulus system concept in ancient Chinese architecture. For example, the *cai* is used as a basic modulus to specify the proportion and size of the wood building component, which in turn defines the shape and modulus of main building components (Li, Andrew i-Kang 2001). Another modulus-system-related example in this book is the height of a bracket, which was the timber module for measurement in the Song Dynasty and graded into eight classes that approximately follow a mathematical progression (Li, Jie 1100). Correspondingly, the timber module was the width of a bracket during the Qing Dynasty and graded into eleven classes. The grading system follows a strict mathematical progression, according to the *Gongcheng Zuofa Zeli* (Qing Dynasty Architecture Method) (Qing Ministry of Works 1734). In this second rulebook of the Qing Dynasty, the concept of *dou kou* was developed and used in its modular system. The standard length of *dou kou* determines the dimensions of all other timber structures (Li, Di 2016). In recent years, a few important research works have used different methods to explore the fundamental concepts of modularization and logical (e.g., mathematical, functional, geometrical) relationships hidden in these rule-based classic architecture books. With the development of digital techniques in architecture, some researchers used shape grammar concepts and parametric techniques to characterize the rules, ratios, and principles in traditional Chinese architecture (Li, Andrew i-Kang 2001, Li, Di 2013, Li, Di 2016; Wu 2005). Though combining the field measurement of extant buildings, Wang et al. (2011) adopted the Western classic architecture principles to analyze the proportions and scales in traditional Chinese architecture, in terms of floor plans, sections, elevations, decorations, etc.

With similar overall purposes, the focus of the present research is on the curvilinear roofs of ancient Chinese wooden architecture. Chinese curvilinear roofs are unique in terms of their functional, hierarchical, and aesthetic features, distinguishing them from other traditional building components. The building method and procedure for these roofs are documented in the *Yingzao fashi* of the Song Dynasty. Due to the need to conserve cultural heritage as evidenced in architecture, architects and scholars have expended significant effort to document and explore the related design and building methods. The most significant work about these traditional curvilinear roofs in modern times is the *Annotated yingzao fashi* by Liang (1983), which explains in depth the *Yingzao fashi*'s design and building rules and adds helpful drawings to illustrate the implementation of these rules. However, actual applications in contemporary architecture are still challenging because of two significant issues. First, text-based descriptions and specific drawing samples for design and building rules are not compatible with computer-aided design. This issue has prompted a number of architectural studies on shape grammar, extracting grammatical rules and principles for historical architectural forms, shapes, and compositions (Chiou and Krishnamurti 1995; Chiou 1996; Li, Andrew i-Kang 2001; Wu 2005; Liu and Wu 2015; Sung 2006). Second, the parameterization of curvilinear roof systems is a complicated procedure in which simple mathematical expressions may not be appropriate. This might be why Liang only added more drawing samples and translated the original languages into the current one without establishing general rules or principles of extraction. This hinders the potential implementation of these unique ancient roofs in historical building renovation and restoration projects, as well as in contemporary buildings with historical features. In the work of Li, Di (2016), a set of 1:1 models of traditional Chinese joint and roof constructed and exhibited in the Venice Biennale 2014 is documented and analyzed. Although the roof models provide some understandings about the proportions and scales of these roofs, such physical construction methods could not specify the roof design variations (Li, Di 2016). In other words, the digitalization of these design and building methods and rules would be significant but has yet to be achieved. Therefore, the central objective of this present research is to parameterize the design and building rules of traditional Chinese curvilinear roofs as stipulated in the *Yingzao fashi*.

To do so, the first step is to illustrate the major characteristics of curvilinear roofs relative to other traditional Chinese building roofs. Next, we

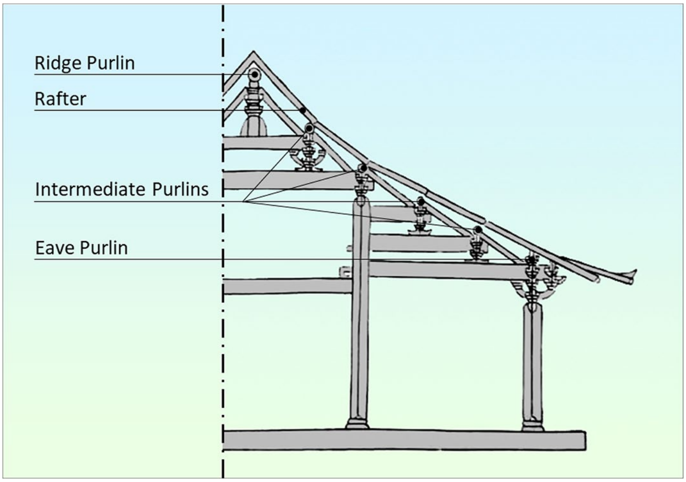

Fig. 1 *Yingzao fashi* Chapter 31:8. Image: Julian Wang and Enhe Zhang, after (Li, Jie 1100)

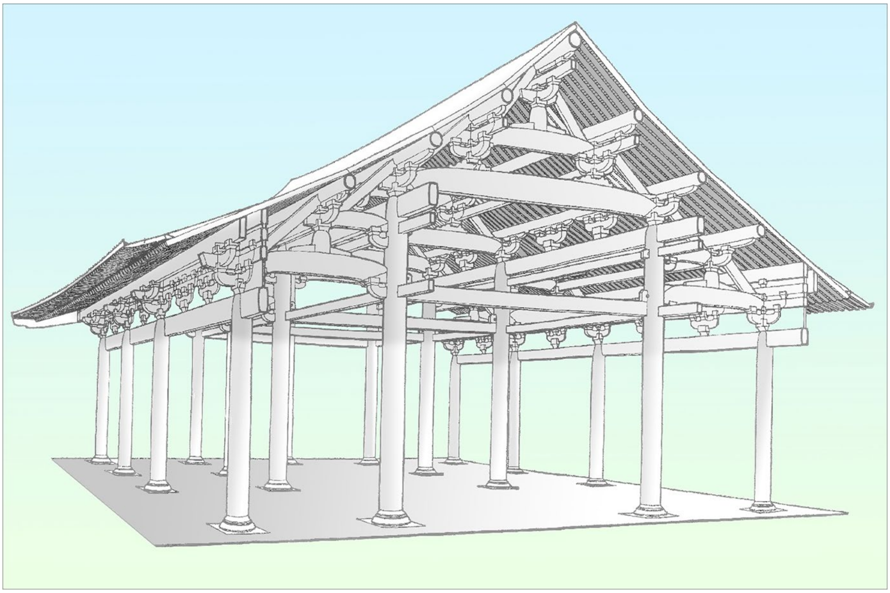

Fig. 2 Song wooden building structure example. Image: Julian Wang, after (Liang 1983)

comprehensively investigate the design and building rules of the "raising and bending system" via the original texts and drawing samples provided in the *Yingzao fashi* and *Annotated yingzao fashi* (Figs. 1, 2). Then, the mathematical expressions for these roof geometry designs and their building rules are developed based on our understanding of the related texts and drawing samples. Newly generated designs based on the mathematical expressions developed are compared to the original graphic information in the *Yingzao fashi,* demonstrating the accuracy of the parametric models. We also perform a brief sensitivity analysis of the mathematical models' parameters. The unique contribution of this research is the fundamental and rational understanding it provides of the design and building rules for this specific roof design for ancient Chinese architecture, enabling generative design by providing the underlying mathematical and design computation procedures.

Characteristics of Traditional Chinese Curvilinear Roofs

The curvilinear roof is often deemed the best character-defining feature of traditional Chinese architecture, an architectural form representing classical Chinese culture. Huiyin Lin once claimed, "it is the roof that has earned the greatest honor for Chinese architecture" (Liang 2001: 13). Itō Chūta, one of the earliest historians to write about Chinese architectural history, argued that aesthetics is one of the primary reasons in China for choosing curvilinear rather than straight pitched roofs (Chūta 2014). Since their initial development, these roofs have been an outstanding feature in Chinese architecture. The original word for "building" (*Wu* in Chinese) actually means "roof," and only now refers to the entirety of the structure. Both in paintings and real-world experience, the roof makes up a significant proportion of a building. It defines the outline, with subtle and graceful variations. Visually, the curved roof implicitly alludes to birds. One of the poems in the *Classic of Poetry,* the oldest existing collection of Chinese poetry dating from the eleventh to seventh centuries BC, describes traditional curved roofs as flowing "Like a bird that has changed its feathers/like a pheasant on flying wings" (Qu 2016). Overhanging gable roofs, light and dynamic, almost appear to float in the air. They dramatically weaken the dense mass standing on the ground and blur the scale of a building. They embody the ponderous cuboid volume in its lively surroundings.

There are various types of curvilinear roofs, with variations in construction method and form. Figure 3 shows the basic representative roof architecture and their hierarchical levels.

In ancient China roof architecture indicated different levels of importance for buildings; in other words, the roof of a traditional Chinese structure was a hierarchical symbol. In particular, there are four basic roof types consisting of hip roof, gable and hip roof, overhanging gable roof, and flush gable roof. The hip roof, also called a *Wudian* roof, divided by one main ridge and four vertical ridges into four slopes, could only be used for imperial palaces and temples; it represented the highest grade of buildings. Gable and hip roofs are a combination of hip and overhanging gable roof components. They consist of one main, four vertical, and four diagonal ridges. This is also called a resting hill, or *Xieshan* roof in the Chinese language, and represents the second level in the hierarchy of importance among examples of ancient Chinese roof architecture. This type was mainly adopted by administrative or ritual buildings such as halls, temples, gardens, and other formal structures. Overhanging gable roofs, also called *Xuanshan* roofs or hanging hill roofs, are two-slope roofs with gables

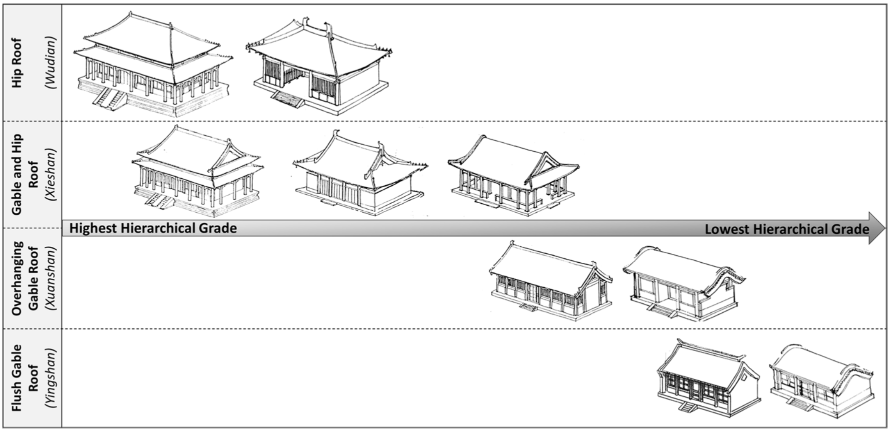

Fig. 3 Basic roof types in traditional Chinese architecture. Image: Julian Wang, after (Liu 2000)

hanging at both ends to protect the sidewalls from rainwater. They are the third grade of traditional Chinese roofs and one of the most frequently used designs for high-level residential buildings and some low-level official buildings in China. A flush gable roof (also called a *Yingshan* roof), is a derivative version of the overhanging gable roof made possible by the massive production of bricks. The sidewalls are sealed above or level with the gables. This simple style of roof was the lowest hierarchical grade roof architecture in ancient China, frequently used in civic buildings. There were other roof shapes with double eaves, which are typical upgraded versions of the associated basic roofs and demonstrate a higher grade in that roof architecture.

In addition to the aesthetic and hierarchical features of traditional Chinese roof architecture, some historical books and documents have also mentioned particular functions for these curvilinear roofs. In "Craftsmen' Records of the Rites of Zhou", compiled towards the end of the Spring and Autumn Period in the fifth century BC: "the hub should be high and the ends of the spokes should be low so that rainwater flows off quickly and splashes away" (Wen 2008). This describes the two main functional demands for curved roofs. First, the curve should let rainwater quickly run over the surface of the roof and scatter onto the ground to reduce the load on the roof. Second, the curve should guide rainwater to a relatively far distance away from the structure so that it splashes away from the walls, preventing damage from the moisture. Therefore, one of the main reasons to adopt this complex method of roof construction (relative to simpler sloped roofs with rigid lines) is related to the rapid discharge of rainwater. It is also worth mentioning, as described by Vitruvius, that the roof design of Roman temples also takes the discharge of rainwater into consideration, in which the roof's sloping and projecting were intentionally designed to cast away rainwater run-off from the exterior walls to avoid staining or damaging them (Vitruvius 1914).

Raising and Bending Rules of Roof Architecture

Texts and Drawing Examples in Ancient Chinese Books of Architecture

It is difficult for wooden structures to be preserved for long periods of time, and hence the number of wooden buildings in China's history is not large, except for some official wooden building groups that were relatively well conserved during the Qing Dynasty. Therefore, the study of historical construction methods related to ancient Chinese architecture relies primarily on books and literature archives. Many ancient Chinese architectural books have two major components: text and drawings used by architects and engineers to design and construct buildings. The texts of these documents include a series of steps, actions, explanations, and some measurement and calculation methods. There are two major ancient Chinese architecture books that are widely recognized in this domain, the *Yingzao fashi* from the Song Dynasty and Qing Dynasty Architecture Method from the Qing Dynasty. These two books recorded all kinds of spatial modeling, using specific scale values and calculation methods. Such calculations appear more frequently in the Qing Dynasty Architecture Method. More calculations and mathematical explorations with detailed drawing samples and photos can be found in Liang's monographs—*Annotated yingzao fashi* and Qing Architecture Regulation (or called *Qingshi Yingzao Zeli* in Chinese) (Liang 2001). Some of the calculations were described in rhyme to make it easier for craftsmen to memorize, master, and communicate with others. Some are compiled into mathematical formulas. However, most of the calculations described emphasize the practicability rather than the fundamental mathematical expression, only introducing key representative parameters, values, and relationships. This is different from the mathematical models and algorithms we see and use in computational design platforms today. Thus, it is challenging for contemporary architects, scholars, and engineers to fully understand and use these calculations. The inclusion of sample drawings is another major characteristic of these books. A set of drawings with narrative text is often used to illustrate an entire construction process: the so-called shape-finding method. Yet there is a substantial difference between the ancient Chinese drawing method and general modern geometric drawing due to the limitations of the drawing techniques and concepts at a certain time. Some effort is necessary to use modern descriptive geometry to redraw what are at times unintelligible drawing examples and convert them into standard engineering drawings for the present age.

Raising and Bending Rules

The curvilinear roofs of ancient Chinese architecture consist of several polylines. Each polyline's generation is based on the raising and bending rules defined in the *Yingzao fashi* and re-introduced in the Qing Dynasty Architecture Method. However, Liang (2001) found the description of this rule in the latter to be more conclusive and oversimplified, without the procedural details and flexibilities introduced in the original record. Therefore, this present study adopted the original texts and drawing samples provided in the *Yingzao fashi* and *Annotated yingzao fashi*. There are five basic types building roofs that were divided into four grades of building in the raising and bending system: Class I, palace-type structure; Class II, mansion-type structure with semi-circular roof tiles; Class III, porch-type structure with semi-circular roof tiles and, mansion-type structure with flat roof tiles; Class IV, porch-type structure with flat roof tiles. We translated the ancient Chinese description into the current language, along with some geometric parameters to make these designs and building rules more readable and convenient to understand.

– *The rule of raising*: First, measure the span (B) between the centerlines of the front and rear eave purlins and divide it into three equal sections. Then, raise the ridge purlin to the heights of 1/3 B, 1/4 B, 1/4 B plus 8/100 B, 1/4 B plus 5/100 B, and 1/4 B plus 3/100 B for Roof Classes I, II, III, and IV, respectively (measuring from the top of the eave purlin to that of the ridge purlin).
– *The rule of bending*: Based on the resultant elevation (H_0) of the ridge purlin from the above-described raising rule, drop the following purlin by 1/10 H_0. Subsequently, the lower amount should decrease by half for each consecutive purlin from the top. For instance, if H_0 equals 200 *cun* (ancient Chinese unit of length), draw a line connecting the center point of the ridge purlin's top and that of the eave purlin's top, and then drop the purlin at the first seam by 20 *cun*. Next, draw another line from the center of the first seam purlin's top to that of the eave purlin's top, and then drop the purlin at the second seam by 10 *cun*. Repeat this process for each consecutive purlin at each seam (e.g., 5 *cun* for the third, 2.5 *cun* for the fourth, etc.). If the span between the purlins is not distributed equally, the lowered amount should be adjusted based on the purlins' variation for each purlin at each seam.

We redrew the geometric schematics shown in Fig. 4 with mathematical expressions and symbols based on the *Annotated yingzao fashi*. Note that we found and corrected an error in the formula for the raising process used in his drawings. In the textual description of the raising and bending rules above and graphic description in Fig. 4, each roof segment's slope is dependent on the overall raising height and previous segment, which informs some of the underlying mathematical relationships and algorithms.

Parameterizing the Raising and Bending Rules

Parametric Model Development

We first choose the palace-type structure to develop a parametric model for the rules for raising and bending when generating curvilinear roofs. As shown in Fig. 5, we set the span between the centerlines of the eave purlin at the front and back as B.

We define the height between each purlin and eave purlin as the number sequence $\{H_n\}$; thus, the initial height H_0 is the raising height for the palace-type

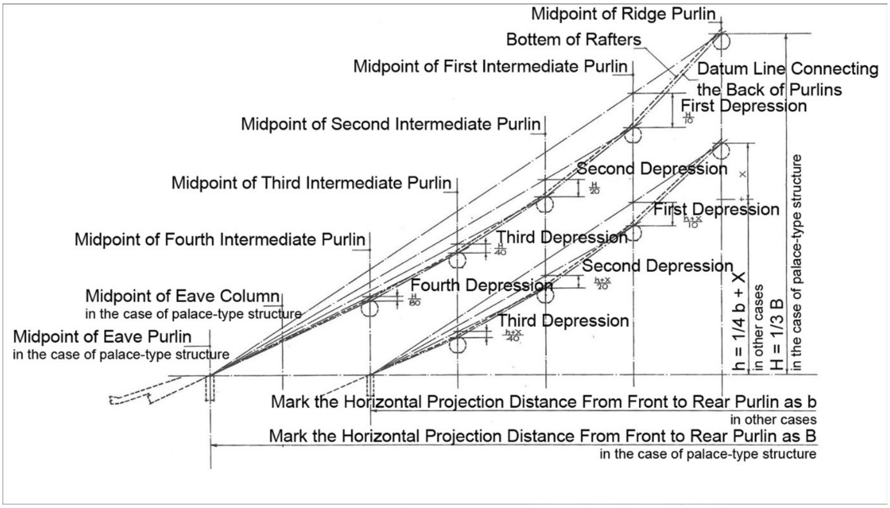

Fig. 4 Design drawings for the rules for raising and bending. Image: Yuan Shen and Enhe Zhang, after (Liang 1983)

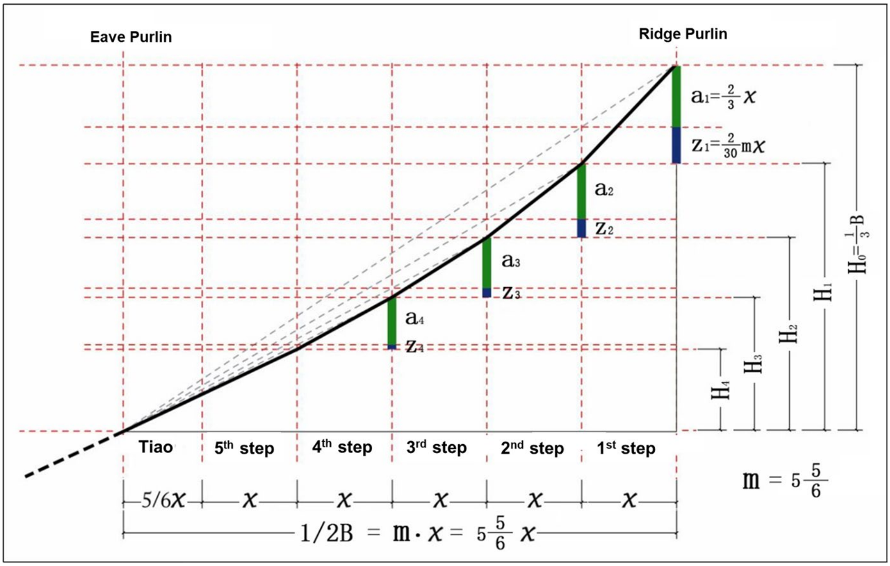

Fig. 5 Variables for the design rules for raising and bending. Image: Yuan Shen and Enhe Zhang

roof,$H_0 = \frac{1}{3}B$. We then set the length of each "rafter span" (i.e., the span between seams) as x, which determines the total step coefficient m. The first step begins from the midpoint of the ridge purlin and ran to the first seam. Note that m is not always an integer because the last eave-overhang section is determined by the tiers. For instance, in palace-type roofs, m equals $5\,\frac{5}{6}$. We are then able to determine that

$B = 2\ m \cdot x$. That gives us the following relationship between the step coefficient and raising height:

$$\frac{1}{2}B = \frac{3}{2}H_0 = m \times x \tag{1}$$

To obtain the mathematical expression of the bending process, we introduce two series, $\{a_n\}$ and $\{z_n\}$. The sum of a_n and z_n represent the difference between consecutive purlin heights. The variable z_n is key to determining the bending levels because it reflects the primary design rule of bending in the *Yingzao fashi*: "the lower amount decreases by half for each consecutive purlin from the top." According to the *Yingzao fashi*'s bending rule, we obtained:

$$a_n = \frac{H_{n-1}}{m-n+1} \tag{2}$$

$$z_n = \left(\frac{1}{2}\right)^{n-1} \times z_1 \tag{3}$$

where $a_1 = \frac{H_0}{m} = \frac{2}{3}x$ and $z_1 = \frac{H_0}{10}$, and n is the positive integer between [1, *m*].

Then, the height series of the purlins $\{H_n\}$ could be expressed as:

$$H_n = H_{n-1} - (z_n + a_n) \tag{4}$$

After applying Eqs. (2) and (3), we obtain:

$$H_n = H_{n-1} - \left(\left(\frac{1}{2}\right)^{n-1} \times z_1 + \frac{H_{n-1}}{m-n+1}\right) = \left(\frac{m-n}{m-n+1}\right)H_{n-1} - \left(\frac{1}{2}\right)^{n-1} \times \frac{1}{10}H_0, \tag{5}$$

where $H_0 = \frac{1}{3}B = mx$, *n [1, m]* and $n\ Z^+$, *m* is the step coefficient, and *x* is the step length.

Generic Parametric Models for Curvilinear Roofs as Defined in the Yingzao fashi

As described in the *Yingzao fashi*, we calculate H_0 for all grades of the roof. To have a more generic expression, we introduce three other parameters—*c*, *h*, and *X*—to calculate the overall raising height. The variable c is the roof type coefficient as defined by the roof grade, h refers to the basic height upon *B*, and *X* represents the adjusted amount for the different grades of roof. The sum of *X* and *h* equals H_0. For example, the raising height for the roof in a mansion-type structure with semi-circular tiles is:

$$H_0 = h + X = h + \frac{8}{100}h = \left(1 + \frac{8}{100}\right)h = \left(1 + \frac{8}{100}\right) * \frac{1}{4}B = \frac{108}{400}B \tag{6}$$

The mathematical models for the raising heights of all four classes of roofs are provided in Table 1. Based on the raising height H_0 for each class of roof, we calculate the key bending parameters $\{z_n\}$ to complete the bending procedure.

Table 1 Parametric Models and Key Variables for the Roof Raising Rule as Defined by the *Yingzao fashi*

Grade	c	h	$X\ (X = \frac{c}{100}h)$	*RaisingheightH*$_0$ ($H_0 = h + X$)
Class I, palace-type structure	$c = 0$	$h = \frac{1}{3}B$	$X = 0$	$H_0 = \frac{1}{3}B$
Class II, mansion-type structure with semi-circular roof tiles	$c = 8$	$h = \frac{1}{4}B$	$X = \frac{8}{100}h$	$H_0 = \frac{108}{400}B$
Class III, porch-type structure with semi-circular roof tiles and mansion-type structure with flat roof tiles	$c = 5$	$h = \frac{1}{4}B$	$X = \frac{5}{100}h$	$H_0 = \frac{105}{400}B$
Class IV, porch-type structure with flat roof tiles	$c = 3$	$h = \frac{1}{4}B$	$X = \frac{3}{100}h$	$H_0 = \frac{103}{400}B$

The parametric models of the bending parameters for all grades of the roof are presented in Table 2. Combining the models in Tables 1 and 2, all height parameters $\{H_n\}$ for the purlins could be obtained by means of Eq. (5), thus generating accurate roof shapes and structures. Note that the models presented in the tables refer specifically to the four roof classes indicated in the *Yingzao fashi*, so some procedure variables (e.g., x, m) are omitted from the tables.

The parametric diagram in Fig. 6 illustrates all of the parameters involved in the parametric models, and it also presents the parameters' relationships and their computation procedures.

Model Validation

As there are no other mathematical models for outputting curvilinear roofs that are available for us to use in validating the models developed here, we chose to compare our models with the drawing samples provided by Liang in the *Annotated yingzao fashi,* based on the textural description and information in the *Yingzao fashi* and according to the principle of descriptive geometry. We generate all four classes of the roof using the parametric models developed here and compared them with the graphic drawing samples supplied in the *Yingzao fashi*. Two key parameters—the raising height of each purlin and the steepness (i.e., the degree of bending)—are compared. The comparison, shown in Fig. 7, demonstrates the agreement between the *Yingzao fashi*'s drawing samples and the generated curvilinear roofs.

As mentioned, there is an error in the raising formula in the original drawing sample in the *Annotated yingzao fashi*. The expression $X = cB/100$ should be $X = ch/100$. As described in *Annotated yingzao fashi*, in ancient Chinese architecture, the overall steepness of a curvilinear roof differs according to the building grade. Compared to lower-grade buildings and roof types, the overall slopes of the roofs in higher-grade buildings are larger, indicating the importance of such buildings in the hierarchy. Therefore, it is necessary to correct this mistake in establishing specific curvilinear roofs as defined in the *Yingzao fashi*; otherwise, the curves generated for

Table 2 Parametric Models and Key Variables for the Roof Bending Rule as Defined by the *Yingzao fashi*

Grade	H_0	Z_1	Bending level $\{z_n\}$
Class I, palace-type structure	$H_0 = \frac{1}{3}B$	$z_1 = \frac{1}{30} \cdot B$	$z_n = \left(\frac{1}{2}\right)^n \cdot z_1$ where $n \in [1, m]$ and $n \in Z^+$, $2mx = B$, and x is rafter span
Class II, mansion-type structure with semi-circular roof tiles	$H_0 = \frac{108}{400}B$	$z_1 = \frac{108}{4000}B$	
Class III, porch-type structure with semi-circular roof tiles and mansion-type structure with flat roof tiles	$H_0 = \frac{105}{400}B$	$z_1 = \frac{105}{4000}B$	
Class IV, porch-type structure with flat roof tiles	$H_0 = \frac{103}{400}B$	$z_1 = \frac{103}{4000}B$	

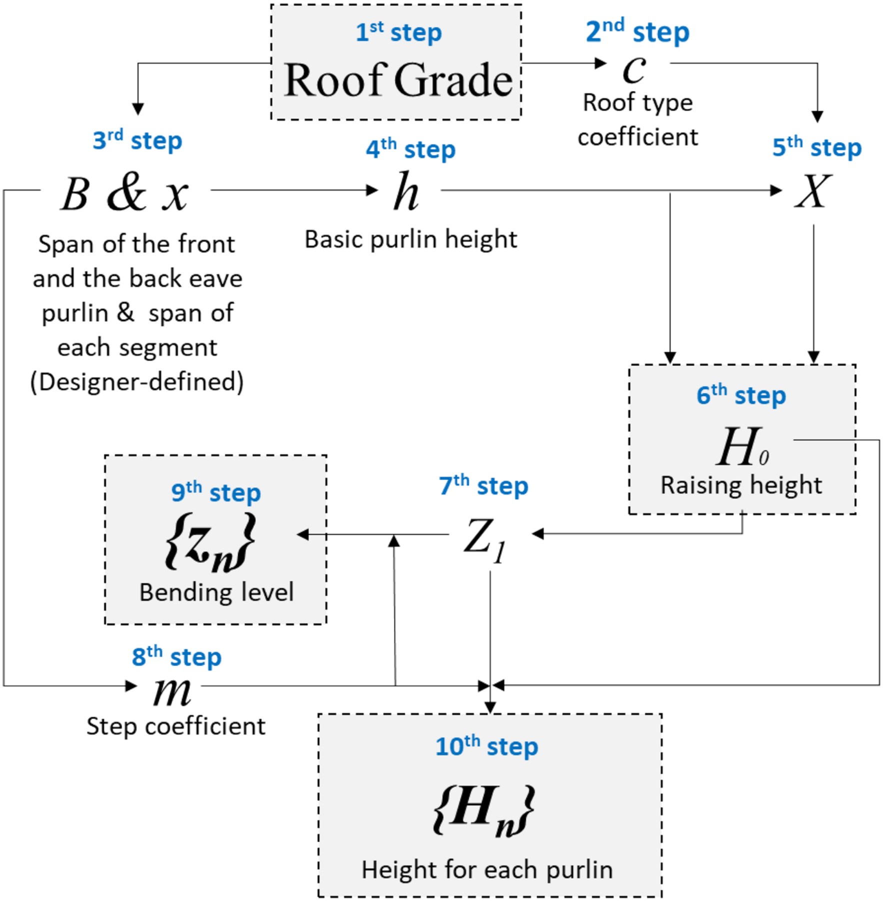

Fig. 6 Map of parametric data flow and relationships. Image: Julian Wang

different classes of roof would be more parallel and without notable differences in steepness.

Discussion

A mathematical algorithm for generating roof curves is developed for the present research. The mathematical models of $\{H_n\}$ and certain user-defined parameters (m and x) can be used to obtain the height of each purlin, given the building and roof classes. Table 3 lists the first three height values for each building grade.

Although there are ten parameters for the entire generation process, only some may have significant effects on determining the resultant roof curves. One of the key parameters is the step coefficient m, which resulted from the user-defined variables B and x, representing the overall span of the back and front purlin each purlin section,

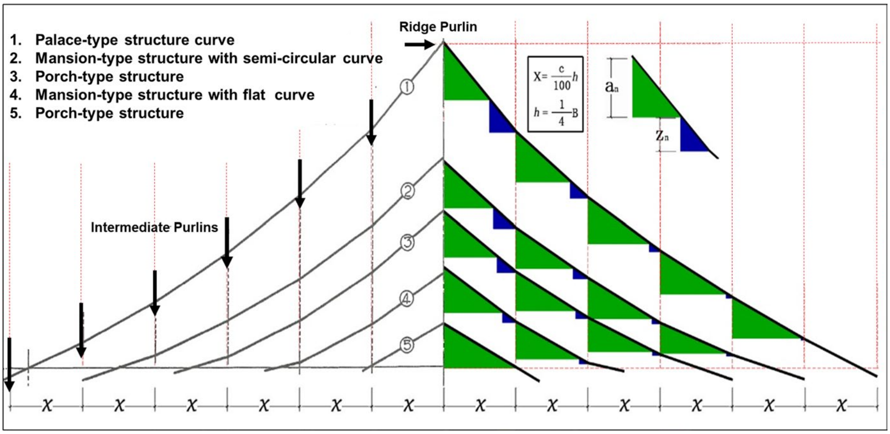

Fig. 7 Comparison of the curvilinear roofs drawn by Liang in the Annotated *yingzao fashi* (left side) and those generated by the parametric models developed in this work (right side). Image: Yuan Shen and Enhe Zhang

Table 3 Key parameters and parametric models by roof class

Grade	Raising height $\boldsymbol{H}_0$ for the ridge purlin	Raising height for the other intermediate purlin $\{\boldsymbol{H}_n\}$ $\boldsymbol{H}_n = \left(\frac{\boldsymbol{m}-\boldsymbol{n}}{\boldsymbol{m}-\mathrm{n}+1}\right)\boldsymbol{H}_{\mathrm{n}-1} - \left(\frac{1}{2}\right)^{\mathrm{n}-1} \cdot \frac{1}{10}\boldsymbol{H}_0$; $\boldsymbol{B} = 2 * (\boldsymbol{m} * \boldsymbol{x})$; where $n \in [1, m]$ and $n\ Z^+$, $2\boldsymbol{mx}=B$, and $\boldsymbol{x}$ is rafter span
Class I	$H_0 = \frac{2}{3} \cdot mx$	$H_1 = \left(\frac{9m-10}{15}\right) \cdot x;\ H_2 = \left[\frac{17m^2-55m+40}{30\cdot(m-1)}\right] \cdot x$
Class II	$H_0 = \frac{108}{200} \cdot mx$	$H_1 = \left(\frac{243m-270}{500}\right) \cdot x;\ H_2 = \left[\frac{459m^2-1512m+1107}{1000\cdot(m-1)}\right] \cdot x \ \ldots$
Class III	$H_0 = \frac{105}{200} \cdot mx$	$H_1 = \left(\frac{189m-210}{400}\right) \cdot x;\ H_2 = \left[\frac{71m^2-231m+168}{160\cdot(m-1)}\right] \cdot x$
Class IV	$H_0 = \frac{103}{200} \cdot mx$	$H_1 = \left(\frac{827m-1030}{2000}\right) \cdot x;\ H_2 = \left[\frac{1651m^2-2581m+2060}{4000\cdot(m-1)}\right] \cdot x$

respectively. The *Yingzao fashi* indicates how many purlins can be used in one building roof. The upper limit of m is 7, so following the ancient Chinese style, at most seven sections between the seams from the eave purlin to the ridge are allowed in any building. This therefore begs the question of why this limit is defined in the *Yingzao fashi*. Is it possible to have higher step coefficient values?

To understand the impact of the step coefficient on the overall roof curve, we made a series of calculations and generated roof curves with different m values. Figure 8 presents the overall roof shape when m equals 100. We find that the roof curve tends to be a straight line after the first seven steps. This echoes the design limit of this parameter in the *Yingzao fashi*.

We also examined the effects of and reason for using 1/2 in the series of z_n, which follows the following design rule in the *Yingzao fashi*: "the lowered amount decreases by half for each consecutive purlin from the top." After analyzing Eq. (3),

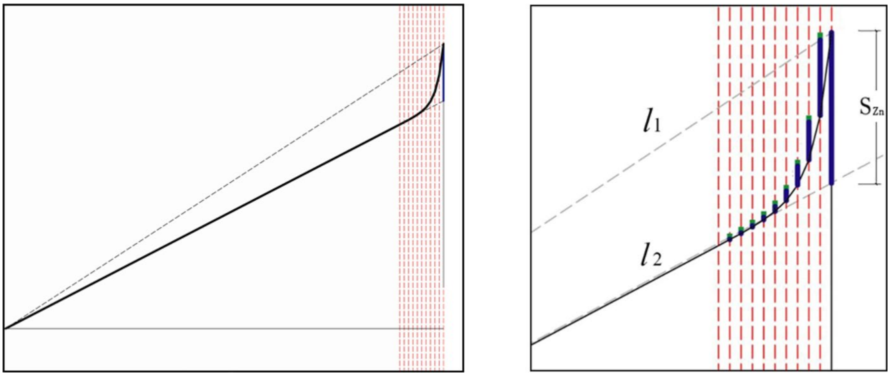

Fig. 8 Curve shape at m = 100 (on the right is the enlarged scope for the initial steps). Image: Yuan Shen

we conclude that z, the bending level, will rapidly converge to zero because of the exponential involvement, and the first several values in $\{z_n\}$ have greater effects on the steepness of the roof curve. The existence of $\{z_n\}$ and its calculation method enable different slope values in a variety of sections (between the seams), forming the Chinese curvilinear roof shape feature. The closer to the roof ridge, the steeper the roof section is. This characteristic is consistent with the traditional Chinese roof design in terms of hierarchy, function, and aesthetics.

Conclusion

This present work adds new insights to existing mathematical methods in studying ancient architecture and construction. In particular, the developed mathematical models of the raising and bending rules are typically sequence-related recursive function systems that are an iterative process where the initial value is given and gives the following or next term by applying the same process repeatedly. In other words, to obtain a raising height value for a specific roof segment by using the recursive process, the previous immediate raising height value should be known. Although the iterative function system has been investigated and applied to some geometric studies (e.g., factal geometry) in architecture, the recursive function system has rarely been discussed in architecture. This work introduces an exemplary case of using the recursive function system in the traditional Chinese roof architecture. The benefit of having this full mathematical expression of the design rules is that more in-depth investigations can now be conducted. For instance, a sensitivity analysis of the key parameters in the iterative mathematical models is conducted in this work, revealing the reasons why certain numbers were selected for these ancient Chinese architecture books. Certainly, there are more comprehensive and deeper topics worth exploring based on these mathematical models. Consider, for example, the discharge of rainwater, which has been mentioned as an essential physical function of traditional curvilinear roofs in the various historical literature about traditional Chinese architecture but has yet to be demonstrated. With the mathematical model developed here, we can accurately derive the functions of the curvilinear shape in a coordinating system and then compare the functions with some known curves, such as the brachistochrone curve. In both mathematics and physics, the brachistochrone curve (or the curve of fastest descent) is the shape that makes an object slide without friction in uniform gravity to a given endpoint in the shortest amount of time. Some physical testing could be also conducted to determine the performance of the rainwater discharge. In addition to this, other possible topics, such as the geometric proportion between the height of the ridge purlin and height of the main inner space determined by the roof's raising and bending rules, wooden structure types under different roof shapes, and generative design procedures that combine curvilinear roofs and base models, can be explored based on the developed mathematical models in the future.

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